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Topological charges of gauge field configurations and order parameter configurations in heterophase systems

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Abstract. Analysis is performed of general features of schemes for calculating the topological charges of heterophase gauge field configurations and order prameter configurations. The feasibility is substantiated of using the results and methods of the theory of topologically non-trivial order parameter configurations for studying complicated heterophase gauge field configurations and vice versa. Structures are revealed of topological charges for string-like configurations of the order parameter which contain ball defects in their cores, as well as for string-like configurations of gauge fields, containing ball monopoles in their cores. Within the framework of the topological approach, string-like defects are studied in superfluid ^{3}He and nematic liquid crystals, as well as string-like configurations of gauge fields in the gauge theory with SU(3) symmetry.

1. Introduction

Gauge field theory is an effective instrument for describing the behaviour of a wide class of physical systems: elementary particle fields (see, e.g., [1, 2]), solids containing dislocations and disclinations [3, 4], amorphous metals [5-7], oxide glasses [8], blue phases of liquid crystals [6], magnetics [4], nematic liquid crystals and superfluid liquids [4, 9, 10]. Special attention is usually paid to the study of topologically non-trivial gauge field configurations (GFC), each representing (according to the topological laws) a stable excited state ('defect') of the physical system described by the gauge theory. Since topologically non-trivial GFC are stable, they are long lived and exert a considerable effect on the behaviour of physical systems.

Topologically non-trivial heterophase GFC are of great interest in Yang-Mills-Higgs gauge theories. Each such configuration is stable and describes the behaviour of a physical system which can be divided into several phases, spatial or spacetime regions characterised by different gauge symmetries. Examples of heterophase GFC are 't Hooft-Polyakov ball monopoles (figure 1) and string-like configurations (figure 2) (see, e.g., [1, 2, 11-13]), as well as GFC of complicated form: walls terminating in strings [14], strings terminating in monopoles [15], loop strings [15], strings with internal ball monopoles [16] and loop monopoles [17].

Related objects for heterophase GFC are heterophase order parameter configurations (OPC) in ordered media with a variable degeneracy space†, namely the space characterising the medium order parameter symmetries (see reviews [18-22]). Each topologically non-trivial heterophase OPC is stable and describes the behaviour of the ordered medium, which can be divided into several phases, namely space regions characterised

† The degeneracy space is often also called the order parameter space or the manifold of internal states.

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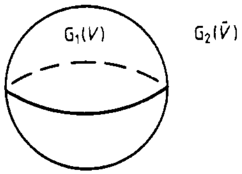

Figure 1. The two-phase configuration which is the 't Hooft-Polyakov monopole (respectively ball defect). The first phase is the ball-like core characterised by a local G_1 symmetry (respectively degeneracy space V). The second phase characterised by a local G_2 symmetry (respectively degeneracy space $\tilde{V}$) occupies the region outside the ball. $G_2 \subset G_1$ ($\tilde{V} \subset V$).

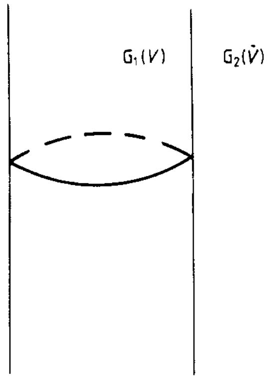

Figure 2. The two phase configuration which is the string-like GFC (line defect). The first phase is the string cylindrical core characterised by a local G_1 symmetry (degeneracy space V). The core is enveloped by the second phase characterised by a local G_2 symmetry (degeneracy space $\tilde{V}$). $G_2 \subset G_1$ ($\tilde{V} \subset V$).

by different degeneracy spaces (DS). As examples of heterophase OPC we can cite string-like and ball defects [23] (figures 1 and 2), surface defects [24], linear and plane solitons [25, 26] and loop defects [27] encountered in magneto-ordered media like liquid crystals and superfluid systems.

The main characteristics of GFC and OPC are their topological charges, which determine the stability of configurations, properties of configurations in case of transformations of their form and phase transitions, laws of merging and separation of configurations [1, 2, 11-13, 18-22]. Calculation methods of topological charges of complicated heterophase OPC are now well developed [20-22], which cannot be said for the topological theory of complicated heterophase GFC.

In the present paper an analysis is performed of general features of schemes for calculating the topological charges of heterophase GFC and OPC, and on the basis of this analysis we establish that the results and methods of the theory of topologically non-trivial heterophase OPC can be used for studying complicated heterophase GFC (section 2). Structures are revealed of topological charges for string-like OPC containing ball defects in their cores, as well as for string-like GFC with intracore ball monopoles (section 3). Topological analysis is performed of string-like OPC in superfluid ^{3}He-A and nematic liquid crystals, as well as of string-like GFC in the gauge theory with broken SU(3) symmetry (section 3).

2. General features of schemes for calculating topological charges of heterophase gauge field configurations and heterophase order parameter configurations

The main intention of the present section is to reveal the general features of schemes of topological classification (determining a set of topological charges) of heterophase GFC and heterophase OPC. Firstly, we shall consider heterophase GFC. For the topological analysis of such configurations it is convenient to use the fibre bundle formalism which is a geometric analogue of gauge theory (see pioneer work [28] and [2, 11-13]).

Within the framework of the geometric approach each GFC is juxtaposed with a fibre bundle, and the problem of classification of GFC is reduced to that of fibre bundles [2, 11-13, 28].

Let us consider the stationary Yang-Mills-Higgs theory with symmetry breakdown of the type $G_1 \to G_2 \to \cdots \to G_n$, where gauge groups G_i are Lie groups and $G_n \subset G_{n-1} \subset \cdots \subset G_1$. In the general case, the heterophase GFC in such a theory describes a physical system, which can be divided into phases which are three-dimensional space regions B_i ($i = 1, \dots, n$) such that (a) the union of the regions B_i is a three-dimensional region B, where the gauge theory is determined: $\bigcup_{i=1}^{n} B_i = B$; (b) the intersection of two regions B_i, B_j is either empty or a two-dimensional region Δ_{ij}; (c) the behaviour of the system in region B_i is characterised by the local G_i symmetry. In accordance with the principles of the geometric interpretation of gauge theory [2, 11, 12, 28], the classification of the above heterophase GFC is identical to that of those principal fibre bundles ξ on base B, which on regions B_i ($B_i \subset B$, $i = 1, \dots, n$) are reduced to fibre bundles $\xi_i(B_i, G_i)$ with the structural group G_i:

$$\xi|_{B_i} = \xi_i(G_i, B_i). \tag{1}$$

For instance, for the 't Hooft-Polyakov monopole (figure 1) the region B represents a three-dimensional Euclidean space $\mathbb{R}^3 = B_1 \cup B_2$ (where B_1 is the ball (the monopole core) and B_2 is the region outside the ball); moreover, the fibre bundle ξ on each region B_i ($i = 1, 2$) represents the G_i bundle $\xi_i(G_i, B_i)$. A similar 'dissection' of the fibre bundle ξ also takes place for the string-like configuration (figure 2), provided $B = \mathcal{R}^3 = B_1 \cup B_2$, where B_1 denotes the infinite cylinder (the string core) and B_2 the region outside the cylinder.

The classification of the principal G bundles on base X ('monphase fibre bundles') is identical to the homotopic classification of maps $X \to B_G$, where B_G is the base of the universal G bundle E_G (for instance, [29-31]). The classification of the fibre bundles ξ 'composed' of several fibre bundles ξ_i, is not reduced, generally speaking, to independent procedures of classifying the fibre bundles ξ_i, i.e. procedures of calculating homotopic classes of maps $B_i \to B_{G_i}$, where B_{G_i} are the bases of universal G_i bundles. Additionally, it is necessary to take into account the conditions imposed by the continuous transition of fibre bundles ξ_i into each other on the boundaries between them, i.e. on the 'interphase' boundaries Δ_{ij}. Taking into account this condition of 'gluing the phases' is the main specific feature of the mathematical scheme for classifying the fibre bundles ξ which correspond to heterophase GFC.

Let us consider now the classification scheme of fibre bundles ξ corresponding to two-phase GFC. On regions B_1, $B_2 \subset B$ each such fibre bundle ξ is reduced to the fibre bundles $\xi_1(G_1, B_1)$ and $\xi_2(G_2. B_2)$, respectively (here B is the base of fibre bundle ξ, $G_2 \subset G_1$). At the boundary Δ_{12} between regions B_1 and B_2 the condition of continuous reduction of the G_1 bundle to the G_2 bundle is satisfied. This condition is equivalent to the existence of a homotopically commutative diagram [29]:

$$\begin{array}{ccc} \Delta_{12} & \xrightarrow{\ t\ } & B_{G_1} \\ & {\scriptstyle \bar{p}}\searrow \quad \nearrow{\scriptstyle t_0} & \\ & B_{G_2} & \end{array} \tag{2}$$

where B_{G_i} are the bases of universal G_i-bundles E_{G_i}, and the map $t_0: B_{G_2} \to B_{G_1}$ is induced by embedding universal fibre bundles: $E_{G_2} \to E_{G_1}$. In this way, for the classification of fibre bundles ξ—calculation of the set A_ξ of equivalence classes of fibre bundles ξ—it is necessary to classify fibre bundles ξ_1, ξ_2 under the condition (2). Since the classification of G_i bundles ξ_i is identical to the homotopic classification of maps

$$f_i : B_i \to B_{\mathrm{G}_i} \tag{3}$$

the set A_ξ is equal to the product of two sets:

$$A = A_{\xi_1} \times A_{\xi_2} \tag{4}$$

where A_{ξ_i} is a set of homotopic classes of maps f_i under condition (2).

In studying heterophase OPC in ordered media, one encounters the problem of the homotopic classification of maps also being like the maps f_i under a condition like (2). So, a two-phase OPC is characterised by the map [27]:

$$g_1 : M_1 \to V \tag{5}$$

which is continuously extended on the map

$$g_2 : M_2 \to \tilde{V}. \tag{6}$$

Here M_1 (M_2) denotes the spatial region occupied by the first (respectively, second) phase of the ordered medium, $V(\tilde{V})$ the degeneracy space characterising internal symmetries of the first (respectively, second) phase, $\tilde{V} \subset V$. For the sake of definiteness, we shall consider M_1, M_2 to be three-phase regions. In order that there be a continuous extension of the map g_1 on the map g_2, it is necessary that there exist the homotopically commutative diagram

$$\begin{array}{ccc} \tilde{\Delta}_{12} & \xrightarrow{\tilde{i}} & V \\ & {\scriptstyle p}\searrow \quad \nearrow{\scriptstyle \tilde{i}_0} & \\ & \tilde{V} & \end{array} \tag{7}$$

where $\tilde{\Delta}_{12}$ is the 'interphase' boundary and embedding $\tilde{i}_0$: $\tilde{V} \to V$ describes the phase transition in the ordered medium.

From the above discussion (see formulae (2), (3), (5)-(7)) there follows a formal equivalence (say, equivalence I) of the problems of topological classifying two-phase GFC and of classifying two-phase OPC. In the case of GFC, phases B_i serve as phases M_i of the ordered medium, while spaces B_{G_1} and B_{G_2} serve as degeneracy spaces V, $\tilde{V}$ of the ordered medium, respectively.

Now we shall introduce the concept of related OPC and GFC. Two-phase OPC and GFC are termed related if the phases M_i of OPC and the phases B_i of GFC are homeomorphic:

$$M_i \overset{\mathrm{hom}}{\sim} B_i \qquad i = 1, 2. \tag{8}$$

For instance, for the 't Hooft-Polyakov monopole the ball defect is the related OPC (see figure 1), while by the related OPC for the string-like GFC we mean the string-like defect (see figure 2). If $M_i \overset{\mathrm{hom}}{\sim} B_i$ ($i = 1, 2$), according to the homotopic topology concepts [30], terms for sets of homotopic classes of the maps $B_i \to B_{\mathrm{G}_i}$ ($i = 1, 2$) satisfying (2) and terms for sets of homotopic classes of maps $M_1 \to V$, $M_2 \to \tilde{V}$ satisfying (7), coincide exactly under the transformations

$$V \leftrightarrow B_{\mathrm{G}_1} \qquad \tilde{V} \leftrightarrow B_{\mathrm{G}_2}. \tag{9}$$

Hence, the structure of the topological charge sets for two-phase GFC coincides exactly with that of OPC (related to the above-mentioned GFC) under the transformations (9).

Equivalence I, the concept of related GFC and OPC, and similarity of topological charge set structures of related GFC and OPC is directly generalised to the case of an arbitrary number of heterophase configuration phases.

Usually (for instance, [23-27, 32, 33]) the sets of topological charges for two-phase OPC—which are the sets of homotopic classes of the maps (5), (6) satisfying (7)—represent cohomology groups, whose coefficients are homotopic groups

$$\Pi_r(V) \qquad \Pi_r(\tilde{V}) \tag{10}$$

or their subgroups of type

$$\begin{aligned} &\mathrm{Im}(\Pi_r(\tilde{V}) \xrightarrow{\tilde{\varphi}_r} \Pi_r(V)) \qquad \mathrm{Ker}(\Pi_r(\tilde{V}) \xrightarrow{\tilde{\varphi}_r} \Pi_r(V)) \\ &\Pi_r(V)/\mathrm{Im}(\Pi_r(\tilde{V}) \xrightarrow{\tilde{\varphi}_r} \Pi_r(V)) \end{aligned} \tag{11}$$

where homomorphisms $\tilde{\varphi}_r$ are induced by embedding $\hat{\lambda}_0$: $\tilde{V} \rightarrow V$. Due to the equivalence I, heterophase GFC are classified by topological charges, which are elements of groups (10), (11), provided (9). To reduce the calculation procedure of such groups to a convenient form we shall use the properties of the following commutative diagram [31], consisting of two exact sequences of homotopic groups and their homomorphisms:

$$\begin{array}{ccccccccc} \rightarrow & \Pi_r(G_2) & \rightarrow & \Pi_r(E_{G_2}) & \rightarrow & \Pi_r(B_{G_2}) & \rightarrow & \Pi_{r-1}(G_2) & \rightarrow \\ & \downarrow{\scriptstyle \theta_r} & & \downarrow & & \downarrow{\scriptstyle \varphi_r} & & \downarrow{\scriptstyle \theta_{r-1}} & \\ & \Pi_r(G_1) & \rightarrow & \Pi_r(E_{G_1}) & \rightarrow & \Pi_r(B_{G_1}) & \rightarrow & \Pi_{r-1}(G_1) & \rightarrow . \end{array} \tag{12}$$

Here homomorphisms φ and θ are induced by the embedding $E_{G_1} \rightarrow E_{G_2}$. Since $\Pi_r(E_{G_1}) = 1$ by definition, then [31]:

$$\Pi_r(B_{G_i}) = \Pi_{r-1}(G_i). \tag{13}$$

From (13) and the condition of commutability of diagram (12), we find

$$\mathrm{Ker}\ \varphi_r = \mathrm{Ker}\ \theta_{r-1} \tag{14}$$

$$\mathrm{Im}\ \varphi_r = \mathrm{Im}\ \theta_{r-1} \tag{15}$$

and hence

$$\Pi_r(B_{G_1})/\mathrm{Im}\ \varphi_r = \Pi_{r-1}(G_1)/\mathrm{Im}\ \theta_{r-1}. \tag{16}$$

Because of the equivalence I, as well as from (9), (13)-(16) it follows that for topological charge sets of two-phase GFC it is feasible to use the formulae determining the topological charge sets of the 'related' (see the condition (8)) two-phase OPC, provided the following change of groups is performed:

$$\Pi_r(V) \rightarrow \Pi_{r-1}(G_1) \qquad \Pi_r(\tilde{V}) \rightarrow \Pi_{r-1}(G_2) \tag{17}$$

and the change of the corresponding homomorphisms of these groups. The formula (17) also directly generalises to the case of n-phase systems ($n \geq 1$).

So, the general calculation scheme of topological charges for heterophase GFC has common features with the topological analysis scheme of heterophase OPC. Hence, it is feasible to make effective use of the methods and results of the theory of topological excitations in ordered media for studying characteristics of topologically non-trivial heterophase GFC, and vice versa.

3. Topological charges of string-like configurations in superfluid ^{3}He–A, biaxial nematic liquid crystals and gauge SU(3) theory

The present section deals with the topological analysis, performed using concepts developed above, of string-like OPC and GFC containing ball excitations in their cores.

Consider a line (string-like) defect with intracore ball defects (figure 3) in a condensed medium characterised by the variable DS which is assumed to vary as follows

$$W \to V \to \tilde{V} \qquad (\tilde{V} \subset V \subset W). \tag{18}$$

The reasons for and examples of such behaviour of the ordered media DS are discussed in [20-22]. A linear defect (figure 3) is a three-phase OPC. The cores of the ball defects are a phase, characterised by the DS W, the string core is characterised by DS V and the region outside the string core is characterised by DS $\tilde{V}$.

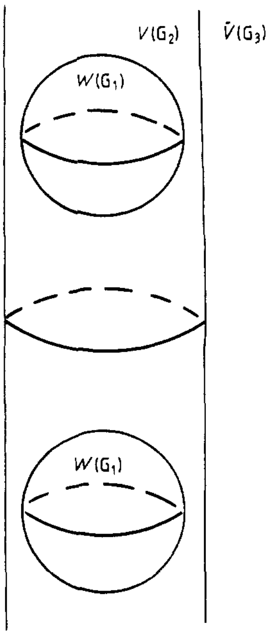

Figure 3. String-like defect (GFC) with intracore ball defects (monopoles), which is a three-phase configuration. W, V, $\tilde{V}$ are degeneracy spaces characterising the order-parameter symmetries of a condensed medium within the ball defect cores, within the string core and outside the string core, respectively. (G_1, G_2, G_3 are gauge groups characterising local symmetries of a system within the ball monopole cores, within the string core and outside the string core, respectively.)

After some analysis, carried out by topological methods developed in [27, 33], we find the topological charge of the line defect with m intracore ball defects to be the set

$$(\tilde{\alpha}, \tilde{\beta}_1, \ldots, \tilde{\beta}_{m+1}, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_m) \tag{19}$$

where the subcharges $\tilde{\alpha}$, $\tilde{\beta}_i$, $\tilde{\gamma}_i$ are elements of the following groups

$$(\tilde{\alpha} \in) \ \mathrm{Ker}(\Pi_1(\tilde{V}) \xrightarrow{\tilde{\varphi}_1} \Pi_1(V)) \tag{20}$$

$$(\tilde{\beta}_i \in) \ \mathrm{Ker}(\Pi_2(V, \tilde{V}) \to \Pi_1(\tilde{V})) = \Pi_2(V)/\mathrm{Im}(\Pi_2(\tilde{V}) \xrightarrow{\tilde{\varphi}_2} \Pi_2(V)) \tag{21}$$

$$(\tilde{\gamma}_i \in) \Pi_3(W)/\mathrm{Im}(\Pi_3(V) \xrightarrow{\phi} \Pi_3(W)) \tag{22}$$

in which case the homeomorphisms $\tilde{\varphi}_r$, ϕ belong to the corresponding exact sequences of homotopy groups and their homeomorphisms, while for the subcharges $\tilde{\beta}_i$ the following additional condition should be satisfied:

$$\tilde{\beta}_i(\tilde{\beta}_{i+1})^{-1} \in \mathrm{Ker}(\Pi_2(V) \to \Pi_2(W)) \qquad i = 1, \ldots, m. \tag{23}$$

Subcharge $\tilde{\alpha}$ describes the features of a linear defect in the ordered medium with DS $\tilde{V}$, inherent in the OPC under consideration. Each subcharge $\tilde{\beta}_i$ characterises the order parameter distribution on that part of the linear defect core situated between the $(i-1)$th and ith ball defects (figure 3). Each subcharge $\tilde{\gamma}_i$ characterises the distribution of the order parameter in the core proper of the ith ball defect.

Now we proceed to discussing the gauge theory with the symmetry breakdown:

$$G_1 \to G_2 \to G_3 \qquad (G_3 \subset G_2 \subset G_1). \tag{24}$$

In such a theory, GFC related to a linear defect with internal ball defects is represented by a string-like GFC containing ball monopoles in its core (figure 3). Such a configuration is a three-phase GFC. The monopole cores are a phase characterised by the gauge G_1 symmetry, the string core by G_2 symmetry, and the region outside the string core by G_3 symmetry (figure 3).

To determine the structure of the topological charge of the GFC in question, we shall use the analogy between the calculation schemes for topological charges of OPC and GFC developed in section 2. In accordance with this analogy, in the case of three-phase systems (ordered media with DS variable as (18), and gauge field systems with the symmetry breakdown (24)) the sets of topological charges of related (see the condition (8)) OPC and GFC coincide exactly under the changes of groups:

$$\Pi_r(W) \leftrightarrow \Pi_{r-1}(G_1) \qquad \Pi_r(V) \leftrightarrow \Pi_{r-1}(G_2) \qquad \Pi_r(\tilde{V}) \leftrightarrow \Pi_{r-1}(G_3) \tag{25}$$

and corresponding changes of the homomorphisms of these groups.

Due to the above analogy between topological analysis schemes for OPC and GFC, one finds the topological charge of string-like GFC with m intracore ball monopoles to be the set

$$(\alpha, \beta_1, \ldots, \beta_{m+1}, \gamma_1, \ldots, \gamma_m) \tag{26}$$

where the subcharges α, β_i, γ_i are defined by the formulae (20)-(23), provided the change (25) comes into play.

In doing so, those properties of a string-like configuration in a system with G_3 gauge symmetry that are inherent to GFC in question are characterised by the subcharge α. Each subcharge β_i describes the gauge field distribution on that part of the string core situated between the $(i-1)$th and ith ball monopoles. The gauge field distribution in the core of the ith monopole is characterised by the subcharge γ_i.

Let us study examples of string-like configurations with intracore ball excitations in condensed media and the gauge SU(3) theory.

(A) *Superfluid ^{3}He-A in a magnetic field.* In discussing a line defect with intracore ball defects one finds that the order-parameter symmetries are different in regions outside and inside the line defect core and inside the ball defect cores. These symmetries are characterised by the following DS

$$\tilde{V} = S^1 \times S^1 \qquad V = SO(3) \qquad W = (S^2 \times SO(3))/Z_2 \tag{27}$$

respectively. Here S^r denotes the r-dimensional sphere, SO(3) the group of three-dimensional proper rotations, $Z_2 = (1, -1)$—the two-element group.

After some calculations using the formulae (19)-(22), (27) we reveal the topological charge of the line defect with m intracore ball defects in superfluid ^{3}He-A (placed in a magnetic field) as being the set $(z_1, \ldots, z_{m+2})$ of integers. In this event the subcharge $\tilde{\alpha} = (z_1, z_2)$ (provided the sum $z_1 + z_2$ is even), each subcharge $\tilde{\gamma}_i = z_{i+2}$ $(i = 1, \ldots, m)$, and the subcharges $\tilde{\beta}_i$ are trivial. (In analogy with particle-like solitons [34], each ball defect characterised by trivial subcharges $\tilde{\beta}_i$ can be unstable with respect to changes in the scales of the defect core. The question of what mechanism causes the stability of the defect scales is beyond the scope of the present paper.)

(B) *Superfluid ^{3}He-A placed in a strong magnetic field.* For a line defect with m internal ball defects the DS characterising the system phases are as follows:

$$\hat{V} = S^1 \times S^2 \qquad V = (S^1 = SO(3))/Z_2 \qquad W = (S^2 \times SO(3))/Z_2. \tag{28}$$

Topological analysis based on the formulae (19)-(22), (28) indicates the topological charge of the discussed defect to be the set $(z_1, \ldots, z_{m+1})$ of integers. Here the subcharge $\tilde{\alpha} = z_1$, the subcharges $\tilde{\gamma}_i = z_{i+1}$ $(i = 1, \ldots, m)$, and $\tilde{\beta}_i$ are trivial.

(C) *Biaxial nematic liquid crystal.* Internal symmetries of biaxial nematics are described by the DS $\tilde{V} = S^3/Q$, where $Q = \{\pm I, \pm i\sigma_x, \pm i\sigma_y, \pm i\sigma_z\}$ is the group of quaternions; I is the unit 2×2 matrix and σ_x, σ_y, σ_z are the Pauli matrices [18-22]. In a biaxial nematic four types of topologically stable line defects/disclinations can exist, which are characterised by non-trivial conjugacy classes of elements of the group Q [18-22]:

$$\bar{C}_0 = \{-I\} \qquad C_x = \{\pm i\sigma_x\} \qquad C_y = \{\pm i\sigma_y\} \qquad C_z = \{\pm i\sigma_z\}. \tag{29}$$

In doing this, C_x and C_y disclinations are line singularities, whereas $\bar{C}_0$ and C_z disclinations have non-singular string-like cores, in which a uniaxial nematic state of the liquid crystal is realised [35, 36], being characterised by the DS $V = S^2/Z_2$. In such string-like cores the experimental evidence [36] indicates that the ball defects are present, the cores of which are characterised by the DS $W = S^4$ [21].

Topological analysis of string-like disclinations with m intracore ball defects reveals the disclination charge to be the set $(\tilde{\alpha}, z_1, \ldots, z_{m+1})$, in which case the subcharge $\tilde{\alpha}$ is $\bar{C}_0$ or C_z, the subcharges $\tilde{\beta}_i$ are integers z_i, and $\tilde{\gamma}_i$ are trivial. An example of the disclination with the topological charge $(\bar{C}_0, 1, 0)$ is pictured in figure 4.

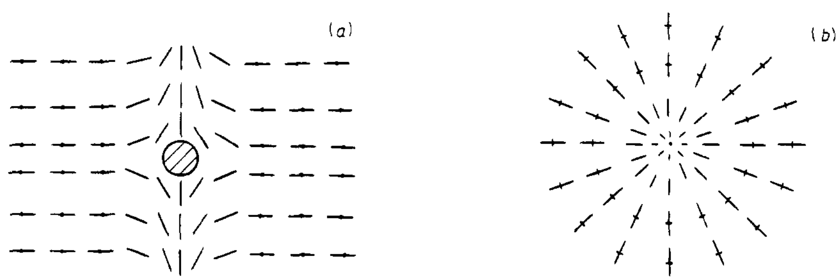

Figure 4. Line disclination with the topological charge $(\bar{C}_0, 0, 1)$ in a biaxial nematic liquid crystal. In the disclination core a ball defect is present. (*a*) Side view. (*b*) Top view.

Let us discuss now heterophase GFC in the theory with the broken gauge symmetry (24), where the group $G_1 = SU(3)$ (by $\lambda_1, \ldots, \lambda_8$ we mean the generators of the group SU(3)), and the group $G_2 = U(2)$ (the generators of the group U(2) are $\lambda_1/2$, $\lambda_2/2$, $\lambda_3/2$, λ_8) while for group G_3 the following subgroups of the group U(2) will be used: the group $U^8(1)$ (the generator λ_8) and the group $U^3(1)$ (the generator $\lambda_3/2$). Detailed consideration of the above gauge theory may be found in [37].

Due to the formula (26), caused by the analogy between schemes for calculating the topological charges of OPC and GFC (see section 2) we obtain after some algebra the topological charge of string-like GFC with m intracore ball monopoles to be the integer set $(z_1, \ldots, z_{m+1})$ in both cases ($G_3 = U^8(1)$ and $C_3 = U^3(1)$). In this event the subcharges $\beta_i = z_i$, while α and γ_i are trivial.

Strings with the trivial subcharge α can be unstable with respect to spreading string cores. The analysis of this problem is beyond the scope of the present paper.

4. Concluding remarks

Thus, the schemes for calculating the topological charges of heterophase OPC and GFC have many similar features. This allows the results of the topological theory of heterophase OPC to be effectively used in studying the topological characteristics of heterophase GFC (given the changes of types (17), (26)) and vice versa. The topological charges for heterophase OPC are defined by the structure of the DS characterising the phases of a condensed medium. The charges of heterophase GFC are caused by the structure of gauge groups describing the local symmetries being inherent to the phases of a gauge field system. Both line defects with intracore ball defects in superfluid ^{3}He–A and string-like GFC with intracore ball monopoles in the SU(3) gauge theory are characterised by topological charges which are sets of integers. For string-like disclinations in biaxial nematic liquid crystals the topological charge consists of integers and the subcharge $\tilde{\alpha}$ which is a non-trivial class of the quaternion group elements.

References

[1] Lopes J L 1983 *Gauge Field Theories* (Oxford: Pergamon)
[2] Konopleva N P and Popov V N 1981 *Gauge Fields* (New York: Harwood)
[3] Kadic A and Edelen D G B 1983 *Gauge Theory of Dislocations and Disclinations* (Berlin: Springer)
[4] Dzyaloshinskii I E 1981 *Physics of Defects* ed R Balian, M Kleman and J P Poirier (Amsterdam: North-Holland) p 317
[5] Nelson D R 1983 *Phys. Rev.* B **28** 5515
[6] Sethna J P 1985 *Phys. Rev.* B **31** 6278
[7] Venkataraman G and Sahoo D 1986 *Contemp. Phys.* **27** 3
[8] Rivier N and Duffy D M 1982 *J. Physique* **43** 293
[9] Sarkar S 1982 *J. Phys. C: Solid State Phys.* **15** 6513
[10] Kawasaki K and Brand H R 1985 *Ann. Phys., NY* **160** 420
[11] Eguchi T, Gilkey P B and Hanson A J 1980 *Phys. Rep.* **66** 213
[12] 1982 *Geometric Techniques in Gauge Theories* (*Lecture Notes in Mathematics* **926**) (Berlin: Springer)
[13] Jaffe A and Taubes C 1980 *Vortices and Monopoles* (Boston: Birkhäüser)
[14] Kibble T W B, Lazardies G and Shafi Q 1982 *Phys. Rev.* D **26** 435
[15] Vilenkin A 1982 *Nucl. Phys.* B **196** 240
[16] Hindmarsh M and Kibble T W B 1986 *Phys. Rev. Lett.* **55** 2398
[17] Ovid'ko I A 1987 *Zh. Exp. Teor. Fiz. Pis. Red.* **45** 3
[18] Mermin N D 1979 *Rev. Mod. Phys.* **51** 591
[19] Michel L 1980 *Rev. Mod. Phys.* **52** 617
[20] Mineev V P 1980 *Sov. Sci. Rev.* A **2** 173
[21] Trebin H R 1982 *Adv. Phys.* **31** 195
[22] Ovid'ko I A and Romanov A E 1987 *Phys. Status Solidi* A **104** 13
[23] Mermin N D, Mineev V P and Volovik G E 1978 *J. Low Temp. Phys.* **33** 117
[24] Volovik G E 1978 *Zh. Exp. Teor. Fiz. Pis. Red.* **28** 65
[25] Mineev V P and Volovik G E 1978 *Phys. Rev.* B **18** 3197

[26] Ovid'ko I A 1985 *Zh. Exp. Teor. Fiz.* **89** 1301
[27] Ovid'ko I A 1986 *Zh. Exp. Teor. Fiz.* **91** 1427
[28] Lubkin E 1963 *Ann. Phys., NY* **23** 233
[29] Husemoller D 1966 *Fibre Bundles* (New York: McGraw-Hill)
[30] Dubrovin B A, Novikov S P and Fomenko A T 1979 *Modern Geometry* (Moscow: Nauka) (in Russian)
[31] Mischchenko A S 1984 *Vector Bundles and their Applications* (Moscow: Nauka) (in Russian)
[32] Ovid'ko I A 1985 *Phys. Lett.* A **110** 408
[33] Ovid'ko I A and Romanov A E 1986 *Commun. Math. Phys.* **105** 443
[34] Volovik G E and Mineev V P 1977 *Zh. Exp. Teor. Fiz.* **73** 767
[35] Kurik M V and Lavrentovich O D 1988 *Usp. Fiz. Nauk* **154** 381
[36] Gálerne Y and Liebert L 1985 *Phys. Rev. Lett.* **55** 2449
[37] Bais F A 1981 *Phys. Lett.* B **98** 437